

# Clock statistics for 1d Schrödinger operators

Victor Chulaevsky <sup>\*</sup>      Fumihiko Nakano <sup>†</sup>

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## Abstract

We study the 1d Schrödinger operators with alloy type random supercritical decaying potential and prove the clock convergence for the local statistics of eigenvalues. We also consider, besides the standard i.i.d. case, more general ones with exponentially decaying correlations.

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## 1 Introduction

The level statistics problem for 1d Schrödinger operators with random decaying potentials were studied by many researchers, and various interesting results have been obtained (cf., e.g., [1], [5], [8], [9], [12], [14]). Usually, one introduces local Hamiltonians  $H_n$  in intervals of size  $n$ , and considers the point process  $\xi_n$  generated by of the suitably rescaled eigenvalues of  $H_n$ . Killip and Stoiciu [5] showed that for CMV matrices weakly  $\xi_n$  converges to the clock process, limit of circular  $\beta$ -ensemble, and the Poisson process for the supercritical, critical, and subcritical cases, respectively. For the Schrödinger operator, similar results are obtained by Avila *et al.* [1] (supercritical discrete model), Kriecherbauer *et al.* [11] (critical discrete model) and by Kotani and Nakano [8], [9], [14] (the continuous model where the random potential is a

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<sup>\*</sup>Département de Mathématiques, Université de Reims, Moulin de la Housse, B.P. 1039 51687 Reims Cedex 2, France. E-mail: victor.tchoulaevski@univ-reims.fr

<sup>†</sup>Department of Mathematics, Gakushuin University, 1-5-1, Mejiro, Toshima-ku, Tokyo, 171-8588, Japan. E-mail : fumihiko@math.gakushuin.ac.jp

function of the Brownian motion on a torus). The aim of the present paper is to prove the clock convergence (i.e., convergence to the clock process) for the Schrödinger operator with the alloy type potential,

$$H = -\frac{d^2}{dt^2} + V(t), \quad V(t) = \sum_{j \in \mathbf{Z}} \frac{\omega(j)}{j^\alpha} f(t-j),$$

where  $\alpha > \frac{1}{2}$ ,  $f \in L^\infty$  with  $\text{supp } f \subset [0, 1]$ , and the amplitudes  $\omega_j$  of the potential on the cells  $[j, j+1)$  are i.i.d. or, more generally, form a stochastic process with exponentially decaying correlations.

Specifically, we consider one of the following two cases.

**A (i.i.d.):**  $\omega(j)$  are i.i.d.,  $|\omega(j)| \leq 1$ , with  $\mathbf{E}[\omega(j)] = 0$  and  $\mathbf{E}[\omega^2(j)] > 0$ . Then by [6],  $H$  has a.s. purely a.c. spectrum on  $[0, \infty)$ .

**B (exponentially decaying correlation):**  $\{\omega(j)\}_{j=1}^\infty$  is a bounded stochastic process such that  $|\omega(j)| \leq 1$ , adapted to a filtration  $\{\mathcal{F}_j\}_{j=1}^\infty$ , with exponentially decaying correlations :

$$|\mathbf{E}[\omega(j)|\mathcal{F}_k]| \leq e^{-\rho|j-k|}, \quad k < j, \quad \rho > 0.$$

Clearly, **A** is obtained as a special case of **B** by setting  $\mathcal{F}_n$  to be the  $\sigma$ -algebra generated by  $\{\omega(j)\}_{j=-\infty}^n$ .

**Remark 1.1** *Examples of **B** are provided, e.g., by the following framework: let  $(\omega, \mathcal{F}, \mathbf{P}, T)$  be an ergodic dynamical system with discrete time  $\mathbf{N}$  or  $\mathbf{Z}$  admitting a finite Markov partition  $\Omega = C_1 \cup \dots \cup C_M$ ,  $M > 1$ . Pick a vector  $f = (f(1), \dots, f(M)) \in \mathbf{R}^M$  and set*

$$F(\theta) := \sum_{i=1}^M f(i) 1_{C_i}(\theta),$$

$$\omega(j) = \omega(j, \theta) := F(T^j \theta),$$

where  $1_{C_i}(\cdot)$  is the indicator function of the partition element  $C_i$ . Then the stochastic process  $\{\omega(j)\}_{j=1}^\infty = \{\omega(j, \theta)\}_{j=1}^\infty$  satisfies **B**. A classical example is given by  $\Omega = \mathbf{T}^1 = \mathbf{R}/\mathbf{Z}$  equipped with the Haar measure, and  $T$  is the dyadic transformation (an endomorphism)  $T : \theta \mapsto \{2\theta\}$  (here  $\{\cdot\}$  stands

for the fractional part). Another well-known example is the so-called natural extension of the dyadic transformation (to an isomorphism), called Baker's transformation (see its definition in Section 7.2.2). Furthermore, there are infinitely many examples given by hyperbolic automorphisms of tori  $\mathbf{T}^\nu$ ,  $\nu \geq 2$  which we also discuss in Section 7.2.3.

In the present paper we focus on the case of the most rapid (exponential) decay of correlations, and stress the fact that the key arguments and estimates rely on bilinear, pair correlations, so they are applicable even to stochastic processes emerging from deterministic dynamical systems, such as mentioned above. There are of course more "stochastic" (viz., non-deterministic) random processes to which our techniques apply, for example, ergodic Markov chains on a finite or countable phase space, with discrete or continuous time. On the other hand, recall that there are uncorrelated stochastic processes which are not i.i.d. For example, one can take the dyadic transformation and set  $\omega(j, \theta) = \cos(2^j \cdot 2\pi\theta)$ ; then  $\mathbf{E}[\omega(j)\omega(k)] = 0$  for  $j < k$  by orthogonality of the standard Fourier basis on  $[0, 2\pi]$ , yet  $\omega(j)$  and  $\omega(k)$  are not independent. Indeed, taking  $j = 1$ ,  $k = 2$ , we have  $\mathbf{E}[\omega^2(1)] > 0$ ,  $\mathbf{E}[\omega(2)] = 0$ , but  $\mathbf{E}[\omega^2(1)\omega(2)] = (2\pi)^{-1} \int_0^{2\pi} \cos^2(\theta) \cos(2\theta) d\theta \neq 0$ .

Finally, note that the condition on the decay rate of pair correlation can be substantially relaxed.

Let  $H_n$  be the Dirichlet restriction  $H|_{[0,n]}$  of  $H$  on  $[0, n]$ , with  $\{E_j(n)\}_{j \geq j_0}$  being its positive eigenvalues, and let  $\kappa_j(n) := \sqrt{E_j(n)}$ . Let  $E_0 > 0$  be the reference energy,  $\kappa_0 := \sqrt{E_0}$ , and consider the point process

$$\xi_n := \sum_{j=1}^{\infty} \delta_{n(\kappa_j(n) - \kappa_0)}.$$

In the case of the free Laplacian, the atoms of  $\xi_n$  are explicitly given by  $\{j\pi - n\kappa_0\}_j$  so that to have the convergence of  $\xi_n$ ,  $n\kappa_0$  needs to converge up to  $\pi$ : we have to consider a suitable subsequence  $\xi_{n_k}$  of such point processes on intervals  $[0, n_k]$ . This is also the case in general which we henceforth assume except Theorem 1.3.

**Assumption S :** A subsequence  $\{n_k\}_{k=1}^{\infty}$  satisfies

$$\kappa_0 n_k = m_k \pi + \beta + o(1), \quad k \rightarrow \infty$$

for some  $m_k \in \mathbf{N}$  and  $\beta \in [0, \pi)$ .

**Theorem 1.1** *Assume **A** and **S**. Then there exists a probability measure  $\mu_\beta$  on  $[0, \pi)$  such that*

$$\lim_{k \rightarrow \infty} \mathbf{E}[e^{-\xi_{n_k}(g)}] = \int_0^\pi d\mu_\beta(\phi) \exp \left( - \sum_{j \in \mathbf{Z}} g(j\pi - \phi) \right), \quad g \in C_c(\mathbf{R}).$$

**Theorem 1.2** *Assume **B** and **S**. Then the statement of Theorem 1.1 remains valid if we take subsequence of  $\xi_{n_k}$  further.*

We believe that the statement of Theorem 1.2 is actually true without taking subsequences. For the moment, the problematic technical issue is the lack of the Burkholder-Davies-Gundy (BDG, in short) type inequality for the models with correlated amplitudes  $\omega(j)$  (cf. Assumption **B**).

Resorting to subsequences is not necessary, however, if we work with another formulation of the problem adopted in [1]. For given  $n$ , rearrange the eigenvalues  $\{\kappa_k(n)\}$  of  $H_n$  in such a way that

$$\cdots < \kappa'_{-1}(n) < \kappa'_0(n) < \kappa_0 < \kappa'_1(n) < \kappa'_2(n) < \cdots$$

Then one has the following result.

**Theorem 1.3 (Strong clock behavior)** *Assume **A**. We then have*

$$(\kappa'_{j+1}(n) - \kappa'_j(n))n \xrightarrow{n \rightarrow \infty} \pi, \quad j \in \mathbf{Z}, \quad a.s.$$

**Remark 1.2** *For the spectral property, the argument in this paper proves the following : (1) In case **A**,  $H$  has purely a.c. spectrum on  $[0, \infty)$  (as is shown in [6]) (2) In case **B**,  $\mu_{ac}(I) > 0$  for any interval  $I(\subset [0, \infty))$ . If BDG inequality were true for case **B**, we would have the same statement as in (1).*

**Remark 1.3** *We can also consider the “decaying coupling constant model” defined as follows.*

$$H'_n := -\frac{d^2}{dt^2} + n^{-\alpha}V(t) \quad \text{on } L^2[0, n]$$

*with Dirichlet boundary condition. Theorems 1.1, 1.2, and 1.3 are also valid for  $H'_n$ , except the fact that  $\phi = \beta$  is deterministic. The proof is simpler : for  $H'_n$  one can show  $n^{-2\alpha}\mathbf{E}[|J^{(n)}|^2] \xrightarrow{n \rightarrow \infty} 0$  by using the method of proof of Proposition 3.1.*

For the proofs of these theorems, we basically follow the strategy of [5, 8, 9] : to study the behavior of the relative Prüfer phase  $\Theta$ . The clock convergence essentially follows from the Hölder continuity of  $\Theta$  with respect to  $\kappa$ , after taking expectation. Assuming **A**, this is done by decomposing  $\Theta$  into the martingale part and the remainder. Assuming **B**, we use the “conditioning argument” used in [3] to prove an extension of the martingale inequality and that of the maximal inequality, which is one of the main ingredient of this paper.

The rest of this paper is organized as follows. In Section 2, we prepare some basic tools such as Prüfer variables and obtain a representation of the Laplace transform of the main point process in terms of the relative Prüfer phase  $\Theta$ , following the argument from [5]. In Section 3, we prove a version of martingale inequality assuming **B**. In Section 4, we prove a version of the maximal inequality using the results in Section 3. In Section 5, we assume **A** and prove the  $p$ -th power version of the results in Section 4, by using the BDG inequality. In Section 6, we prove Theorems 1.1, 1.2, 1.3. In Section 7, a more detailed discussion (continuation to Remark 1.1) is given on dynamical systems satisfying **B**. Throughout this paper,  $C$  stands for a positive constant which may change from line to line in each argument.

## 2 Preliminaries

Let  $H\psi = \kappa^2\psi$ ,  $\psi(0) = 0$ , be a Schrödinger equation on  $[0, +\infty)$  with the Dirichlet condition at 0, which we rewrite as a Cauchy problem for a vector-valued function,

$$\begin{pmatrix} \psi \\ \psi'/\kappa \end{pmatrix} = r_t(\kappa) \begin{pmatrix} \sin \theta_t(\kappa) \\ \cos \theta_t(\kappa) \end{pmatrix}, \quad \theta_t(\kappa) =: \kappa t + \tilde{\theta}_t(\kappa).$$

Then it follows by straightforward calculations that

$$r_t(\kappa) = \exp \left( \frac{1}{2\kappa} \operatorname{Im} \int_0^t V(s) e^{2i\theta_s(\kappa)} ds \right) \quad (2.1)$$

$$\tilde{\theta}_t(\kappa) = \frac{1}{2\kappa} \int_0^t \operatorname{Re}(e^{2i\theta_s(\kappa)} - 1)V(s) \quad (2.2)$$

$$\frac{\partial \theta_t(\kappa)}{\partial \kappa} = \int_0^t \frac{r_s^2}{r_t^2} ds + \frac{1}{2\kappa^2} \int_0^t \frac{r_s^2}{r_t^2} V(s) (1 - \operatorname{Re} e^{2i\theta_s(\kappa)}) ds. \quad (2.3)$$

By Sturm's oscillation theorem,  $j$ -th eigenvalue  $E_j(n)$  of  $H_n$  satisfies  $\theta_n(\sqrt{E_j(n)}) = j\pi$  by which we can derive the following representation of the Laplace transform of  $\xi_L$ .

**Lemma 2.1** *Let*

$$\begin{aligned}\Theta^{(n)}(c) &:= \theta_n(\kappa_c) - \theta_n(\kappa_0), \\ \kappa_c &:= \kappa_0 + \frac{c}{n}, \quad c \in \mathbf{R}\end{aligned}$$

then for  $g \in C_c(\mathbf{R})$ ,

$$\mathbf{E}[e^{-\xi_n(g)}] = \mathbf{E}\left[\exp\left(-\sum_k g\left((\Theta^{(n)})^{-1}(k\pi - \{\theta_n(\kappa_0)\}_{\pi\mathbf{Z}})\right)\right)\right]$$

where

$$\theta_n(\kappa) = [\theta_n(\kappa)]_{\pi}\pi + \{\theta_n(\kappa)\}_{\pi}, \quad [x]_{\pi} := \left\lfloor \frac{x}{\pi} \right\rfloor, \quad \{x\}_{\pi} := x - [x]_{\pi}\pi.$$

By definition,

$$\begin{aligned}\Theta^{(n)}(c) &= c + \tilde{\theta}_n(\kappa_c) - \tilde{\theta}_n(\kappa_0) \\ &= c + \frac{1}{2\kappa_0} \operatorname{Re} \int_0^n (e^{2i\theta_s(\kappa_c)} - e^{2i\theta_s(\kappa_0)}) V(s) ds + O(n^{-1}).\end{aligned}$$

In view of Lemma 2.1, the main task is to show that the 2nd term of RHS tends to 0. To that end, we introduce the functional of the potential

$$J^{(t)}(\kappa) := \int_0^t e^{2i\theta_s(\kappa)} V(s) ds$$

and prove the Hölder continuity of  $J^{(t)}(\kappa)$  with respect to  $\kappa$ . In order to do so, we need the martingale and the maximal inequalities which we establish in the following sections.

### 3 Martingale inequality

The strategy of the proof of martingale inequality in case B is based on a variant of the conditioning employed in [3] and the usual argument to prove the original martingale inequality.

### 3.1 Notation and Statement

In this section, we work under a more general assumption **B** and set

$$J^{(m,N)} := \sum_{j=m}^{N-1} a_j \omega_j, \quad 0 \leq m \leq N$$

for some fixed  $N$ , where  $\{\omega_k\}$  is the stochastic process satisfying the condition **B** and  $a_j$  satisfies a measurability condition :

$$|\mathbf{E}[\omega_j | \mathcal{F}_k]| \leq e^{-\rho|j-k|} \quad (3.1)$$

$$a_j := \frac{b_j}{j^\alpha}, \quad b_j \in \mathcal{F}_{j-c \log j}, \quad c > 0. \quad (3.2)$$

Here we slightly abuse the notation and write  $c \log j$  instead of  $\lfloor c \log j \rfloor$ . The goal of this section is to prove the following propositions.

**Proposition 3.1** *Suppose*

$$|b_j(\omega)| \leq c_j(\omega) j^\eta, \quad \eta \geq 0, \quad \omega \in \Omega$$

with  $\mathbf{E}[|c_j(\omega)|^2] \leq C_{b,\eta}^2$ . Then

$$\mathbf{E}[|J^{(m,N)}|^2] \leq 2C_{b,\eta}^2 \left( \sum_{j=m}^{N-1} \frac{c \log j}{j^{\alpha-\eta}(j-c \log j)^{\alpha-\eta}} + \sum_{j=m}^{N-1} \frac{1}{j^{\alpha-2\eta-1+c\rho}} \right).$$

**Proposition 3.2** *Suppose*

$$|b_j(\omega)| \leq c_j(\omega) j^\eta, \quad \eta \geq 0, \quad \omega \in \Omega.$$

Then

$$\mathbf{E} \left[ \sup_{m \leq n \leq N} |J^{(m,n)}|^p \right]^{1/p} \leq C_1 \left( \mathbf{E}[|J^{(m,N)}|^p]^{1/p} + \mathbf{E}[|D|^p]^{1/p} \right), \quad p > 1.$$

$$\text{where } D := \left( \sum_{j=m}^{N-1} \frac{1}{j^{c\rho+\alpha-\eta}} + \frac{d m^\beta}{m^{\alpha-\eta}} \right) \sup_{m \leq j \leq N} c_j.$$

### 3.2 Proof of Proposition 3.1

We decompose the sum into two parts.

$$\begin{aligned}
\mathbf{E}[|J^{(m,N)}|^2] &= 2 \sum_{m \leq i \leq j \leq N} \mathbf{E}[a_i \omega_i a_j \omega_j] \\
&= 2 \sum_{j=m}^{N-1} \sum_{k=0}^j \mathbf{E}[a_{j-k} \omega_{j-k} a_j \omega_j] \quad (i = j - k) \\
&= 2 \sum_{j=m}^{N-1} \left( \sum_{k=0}^{c \log j} + \sum_{k=c \log j}^j \right) \mathbf{E}[a_{j-k} \omega_{j-k} a_j \omega_j] \\
&=: I + II
\end{aligned}$$

where  $I$  is the sum with  $|i - j| \leq c \log j$ , and  $II$  is the remainder.

$I$  can be estimated easily:

$$\begin{aligned}
I &\leq 2C_{b,\eta}^2 \sum_{j=m}^{N-1} \sum_{k=0}^{c \log j} \frac{1}{j^{\alpha-\eta} (j - c \log j)^{\alpha-\eta}} \\
&\leq 2C_{b,\eta}^2 \sum_{j=m}^{N-1} \frac{c \log j}{j^{\alpha-\eta} (j - c \log j)^{\alpha-\eta}}.
\end{aligned} \tag{3.3}$$

To estimate  $II$ , we use the condition (3.1), (3.2) : for  $k \geq c \log j$  we have  $\omega_{j-k} \in \mathcal{F}_{j-c \log j}$  so that

$$\begin{aligned}
|\mathbf{E}[a_{j-k} a_j \omega_{j-k} \omega_j]| &= |\mathbf{E}[a_{j-k} a_j \omega_{j-k} \mathbf{E}[\omega_j | \mathcal{F}_{j-c \log j}]]| \\
&\leq \mathbf{E}[|a_{j-k} a_j \omega_{j-k}| \cdot |\mathbf{E}[\omega_j | \mathcal{F}_{j-c \log j}]|] \\
&\leq e^{-\rho c \log j} \mathbf{E}[|a_{j-k} a_j \omega_{j-k}|]
\end{aligned}$$

which yields

$$\begin{aligned}
|II| &\leq 2 \sum_{j=m}^{N-1} \sum_{k=c \log j}^j e^{-\rho c \log j} \mathbf{E}[|a_{j-k} a_j \omega_{j-k}|] \\
&\leq 2C_{b,\eta}^2 \sum_{j=m}^{N-1} \frac{1}{j^{\alpha-2\eta-1+c\rho}}.
\end{aligned} \tag{3.4}$$

Using (3.3) and (3.4) completes the proof of Proposition 3.1.  $\square$



### 3.3 Proof of Proposition 3.2

We decompose the sum such as

$$J^{(m,N)} := \sum_{j=m}^{N-1} a_j \omega_j = J^{(m,n)} + J^{(n,N)}, \quad m \leq n \leq N$$

and the 2nd term in RHS is further decomposed into

$$J^{(n,N)} = \sum_{j=n}^{N-1} a_j \omega_j =: J_{A_n} + J_{B_n}$$

$$\text{where } A_n = \{j \geq n \mid j - c \log j \geq n\}, \quad B_n = \{j \geq n \mid j - c \log j \leq n\}.$$

It is easy to see that, for any  $\beta > 0$ , we can find  $d = d_\beta > 0$  such that

$$\#B_n \leq d n^\beta, \quad \forall \beta > 0.$$

(1) For  $J_{A_n}$  we use (3.1), (3.2) :

$$|\mathbf{E}[J_{A_n} | \mathcal{F}_n]| \leq \sum_{j \in A_n} \mathbf{E}[|a_j| \cdot |\mathbf{E}[\omega_j | \mathcal{F}_{j-c \log j}]| | \mathcal{F}_n] = \sum_{j \in A_n} \frac{1}{j^{c\rho+\alpha}} \mathbf{E}[|b_j| | \mathcal{F}_n].$$

(2) For  $J_{B_n}$  we simply use the boundedness of  $\omega_j$  :

$$|\mathbf{E}[J_{B_n} | \mathcal{F}_n]| \leq \sum_{j \in B_n} \frac{1}{j^\alpha} \mathbf{E}[|b_j| | \mathcal{F}_n].$$

Therefore

$$|\mathbf{E}[J^{(n,N)} | \mathcal{F}_n]| \leq \sum_{j \in A_n} \frac{1}{j^{c\rho+\alpha}} \mathbf{E}[|b_j| | \mathcal{F}_n] + \sum_{j \in B_n} \frac{1}{j^\alpha} \mathbf{E}[|b_j| | \mathcal{F}_n].$$

Since  $J^{(m,N)} = J^{(m,n)} + J^{(n,N)}$ ,

$$\begin{aligned} \mathbf{E}[J^{(m,N)} | \mathcal{F}_n] &= J^{(m,n)} + \mathbf{E}[J^{(n,N)} | \mathcal{F}_n] \\ &\geq J^{(m,n)} - \left( \sum_{j \in A_n} \frac{1}{j^{c\rho+\alpha}} \mathbf{E}[|b_j| | \mathcal{F}_n] + \sum_{j \in B_n} \frac{1}{j^\alpha} \mathbf{E}[|b_j| | \mathcal{F}_n] \right). \end{aligned} \quad (3.5)$$

Let

$$T := \begin{cases} \inf \left\{ n \mid m \leq n \leq N, J^{(m,n)} \geq \lambda \right\} & (J^{(m,n)} \geq \lambda \text{ for some } n) \\ N+1 & (\text{otherwise}) \end{cases}$$

Then we have

$$\lambda \mathbf{P} \left( \sup_{n \leq N} J^{(m,n)} > \lambda \right) = \lambda \sum_{n=m}^N \mathbf{P}(T = n) \leq \sum_{n=m}^N \mathbf{E}[J^{(m,n)} ; T = n].$$

Substituting (3.5) yields

$$\begin{aligned} & \lambda \mathbf{P} \left( \sup_{n \leq N} J^{(m,n)} > \lambda \right) \\ & \leq \sum_{n=m}^N \mathbf{E} \left[ \mathbf{E}[J^{(m,N)} | \mathcal{F}_n] + \sum_{j \in A_n} \frac{1}{j^{c\rho+\alpha}} \mathbf{E}[|b_j| | \mathcal{F}_n] + \sum_{j \in B_n} \frac{1}{j^\alpha} \mathbf{E}[|b_j| | \mathcal{F}_n] ; T = n \right] \\ & = \sum_{n=m}^N \mathbf{E}[J^{(m,N)} ; T = n] + \sum_{n=m}^N \sum_{j \in A_n} \frac{1}{j^{c\rho+\alpha}} \mathbf{E}[|b_j| ; T = n] \\ & \quad + \sum_{n=m}^N \sum_{j \in B_n} \frac{1}{j^\alpha} \mathbf{E}[|b_j| ; T = n]. \end{aligned}$$

Let  $A := \{\sup_{n \leq N} J^{(m,n)} > \lambda\}$ . We estimate 2nd and 3rd terms in RHS.

(1) 2nd term :

$$\begin{aligned} & \sum_{n=m}^{N-1} \sum_{j \in A_n} \frac{1}{j^{c\rho}} \mathbf{E}[|a_j| ; T = n] \\ & = \sum_{n=m}^{N-1} \sum_{j=n}^{N-1} \mathbf{1}(n \leq j - c \log j) \frac{1}{j^{c\rho+\alpha}} \mathbf{E}[|b_j| ; T = n, A] \\ & = \sum_{j=m}^{N-1} \frac{1}{j^{c\rho+\alpha}} \sum_{n=m}^{j-c \log j} \mathbf{E}[|b_j| ; T = n, A] \\ & \leq \sum_{j=m}^{N-1} \frac{1}{j^{c\rho+\alpha}} \mathbf{E}[|b_j| ; A] \\ & \leq \left( \sum_{j=m}^{N-1} \frac{1}{j^{c\rho+\alpha-\eta}} \right) \mathbf{E}[\sup_{m \leq j \leq N} c_j ; A]. \end{aligned}$$

(2) 3rd term :

$$\begin{aligned} \sum_{n=m}^{N-1} \sum_{j \in B_n} \frac{1}{j^\alpha} \mathbf{E}[|b_j| ; T = n] & \leq \sup_{n \geq m} \left( \frac{dn^\beta}{n^{\alpha-\eta}} \right) \sum_{n=m}^{N-1} \mathbf{E}[\sup_{j \in B_n} c_j ; T = n, A] \\ & \leq \left( \frac{d m^\beta}{m^{\alpha-\eta}} \right) \mathbf{E}[\sup_{m \leq j \leq N} c_j ; A], \quad \beta < \alpha - \eta. \end{aligned}$$

Therefore

$$\lambda \mathbf{P} \left( \sup_{n \leq N} J^{(m,n)} > \lambda \right) \leq \mathbf{E} \left[ J^{(m,N)} + D; A \right].$$

$D$  is defined in the statement of Proposition 3.2. Let  $\overline{J}^{(m,N)} := \sup_{n \leq N} J^{(m,n)}$ . Then

$$\begin{aligned} \mathbf{E} \left[ |\overline{J}^{(m,N)}|^p \right] &= \int_0^\infty p \lambda^{p-1} \mathbf{P}(A) d\lambda \\ &\leq \int_0^\infty d\lambda p \lambda^{p-2} \int_\Omega d\mathbf{P} 1_A(\omega) \left( J^{(m,N)} + D \right) \\ &\leq \int_\Omega d\mathbf{P} \left( |J^{(m,N)}| + D \right) \int_0^\infty d\lambda p \lambda^{p-2} 1 \left( \lambda < \overline{J}^{(m,N)} \right) \\ &= \frac{p}{p-1} \mathbf{E} \left[ \left( |J^{(m,N)}| + D \right) (\overline{J}_N^{(m,N)})^{p-1} \right] \\ &\leq \frac{p}{p-1} \left( \mathbf{E}[|J^{(m,N)}|^p]^{1/p} + \mathbf{E}[D^p]^{1/p} \right) \cdot \mathbf{E}[(\overline{J}^{(m,N)})^p]^{\frac{p-1}{p}}. \end{aligned}$$

Dividing both sides by  $\mathbf{E}[(\overline{J}^{(m,N)})^p]^{\frac{p-1}{p}}$  completes the proof.  $\square$

## 4 Hölder continuity

In this section we assume  $\mathbf{B}$  and prove a version of the maximal inequalities for  $R$  and  $J$ .

### 4.1 Estimate on $R$

Let

$$\begin{aligned} R^{(m,t)}(\kappa) &:= \int_m^t \left( e^{2i\theta_s(\kappa)} - 1 \right) V(s) ds, \\ R^{(m,n)}(\kappa) &= \sum_{j=m}^{n-1} \frac{\omega(j)}{j^\alpha} \int_j^{j+1} \left( e^{2i\kappa s + 2i\tilde{\theta}_s(\kappa)} - 1 \right) V(s) ds, \quad n \in \mathbf{N}, \\ R^{(n)}(\kappa) &:= R^{(0,n)}(\kappa). \end{aligned}$$

**Proposition 4.1**

$$\begin{aligned}
& \mathbf{E} \left[ \sup_{m \leq t \leq N} |R^{(m,t)}(\kappa)|^2 \right] \\
& \leq C \left\{ \sum_{j=m}^{N-1} \frac{c \log j}{j^\alpha (j - c \log j)^\alpha} + \sum_{j=m}^{N-1} \frac{1}{j^{\alpha-1+c\rho}} + \left( \sum_{j=m}^{N-1} \frac{1}{j^{c\rho+\alpha}} \right)^2 + \left( \frac{d m^\beta}{m^\alpha} \right)^2 \right\} \\
& \quad + C \left( \sum_{j=m}^{N-1} \frac{c \log j}{j^{2\alpha}} \right)^2.
\end{aligned}$$

*Proof.* Fix  $t \leq N$  and set  $n = \lfloor t \rfloor$ . Since

$$R^{(m,t)} = R^{(m,n)} + R^{(n,t)}, \quad |R^{(n,t)}| \leq C n^{-\alpha},$$

it suffices to estimate  $\mathbf{E}[\sup_{m \leq n \leq N} |R^{(m,n)}|^2]$ . Here we set

$$\hat{\theta}_j(\kappa) := \tilde{\theta}_{j-c \log j}(\kappa), \quad c > 0.$$

Then we decompose

$$\begin{aligned}
& R^{(m,n)}(\kappa) \\
& = \sum_{j=m}^{n-1} \frac{\omega(j)}{j^\alpha} \int_j^{j+1} \left\{ e^{2i\kappa s + \hat{\theta}_j(\kappa)} + e^{2i\kappa s + 2i\hat{\theta}_j(\kappa)} \left( e^{2i(\tilde{\theta}_s(\kappa) - \hat{\theta}_j(\kappa))} - 1 \right) - 1 \right\} f(s-j) ds \\
& =: R_1^{(m,n)}(\kappa) + R_2^{(m,n)}(\kappa) + R_3^{(m,n)}(\kappa)
\end{aligned} \tag{4.1}$$

which we estimate separately.

(1)  $R_1$  : Let

$$b_j := \int_j^{j+1} e^{2i\kappa s} f(s-j) ds \cdot e^{2i\hat{\theta}_j(\kappa)}.$$

By Proposition 3.1 ( $\eta = 0$ ),

$$\mathbf{E} [|R_1^{(m,n)}(\kappa)|^2] \leq C \left\{ \sum_{j=m}^{n-1} \frac{c \log j}{j^\alpha (j - c \log j)^\alpha} + \sum_{j=m}^{n-1} \frac{1}{j^{\alpha-1+c\rho}} \right\}.$$

Thus by Proposition 3.2,

$$\begin{aligned}
& \mathbf{E} \left[ \sup_{m \leq n \leq N} |R_1^{(m,n)}(\kappa)|^2 \right] \leq C_1 \mathbf{E} [|R_1^{(m,N)}(\kappa)|^2] + C_2 D^2 \\
& \leq C \left\{ \sum_{j=m}^{N-1} \frac{c \log j}{j^\alpha (j - c \log j)^\alpha} + \sum_{j=m}^{N-1} \frac{1}{j^{\alpha-1+c\rho}} + \left( \sum_{j=m}^{N-1} \frac{1}{j^{c\rho+\alpha}} \right)^2 + \left( \frac{d m^\beta}{m^\alpha} \right)^2 \right\}.
\end{aligned}$$

(2)  $R_2$  : Let

$$b_j = \int_j^{j+1} e^{2i\kappa s + 2i\hat{\theta}_j(\kappa)} \left( e^{2i(\tilde{\theta}_s(\kappa) - \hat{\theta}_j(\kappa))} - 1 \right) f(s - j) ds.$$

Here we estimate

$$\begin{aligned} \left| e^{2i(\tilde{\theta}_s(\kappa) - \hat{\theta}_j(\kappa))} - 1 \right| &\leq 2 \left| \tilde{\theta}_s(\kappa) - \hat{\theta}_j(\kappa) \right| \\ &\leq \frac{2}{2\kappa} \int_{j-c\log j}^{j+1} \left| (e^{2i\theta_u(\kappa)} - 1) V(u) \right| du \\ &\leq \frac{1 + c \log j}{(j - c \log j)^\alpha} \leq \frac{c' \log j}{j^\alpha} \end{aligned}$$

for large  $j$  and for some  $c'$ . Hence we have  $|R_2^{(m,n)}(\kappa)| \leq \sum_{j=m}^{n-1} \frac{c' \log j}{j^{2\alpha}}$  so that

$$\mathbf{E} \left[ \sup_{m \leq n \leq N} |R_2^{(m,n)}(\kappa)|^2 \right] \leq C \left( \sum_{j=m}^{N-1} \frac{c' \log j}{j^{2\alpha}} \right)^2.$$

(3)  $R_3^{(m,n)}$  : this is similar to that for  $R_1^{(m,n)}$ .

$$\begin{aligned} &\mathbf{E}[\sup_{n \leq N} |R_3^{(m,n)}|^2] \\ &\leq C \left( \sum_{j=m}^{N-1} \frac{c \log j}{j^\alpha (j - c \log j)^\alpha} + \sum_{j=m}^{N-1} \frac{1}{j^{\alpha-1+c\rho}} + \left( \sum_{j=m}^{N-1} \frac{1}{j^{c\rho+\alpha}} \right)^2 + \left( \frac{d m^\beta}{m^\alpha} \right)^2 \right). \end{aligned}$$

Putting those estimates together, we complete the proof.  $\square$

## 4.2 Estimate on $J$

For a function  $g = g(\kappa)$  of  $\kappa$ , we set

$$\Delta g := g(\kappa_1) - g(\kappa_2), \quad \kappa_1, \kappa_2 \in \mathbf{R}.$$

The goal of this subsection is to prove the Hölder continuity of  $J$  in the following sense.

**Proposition 4.2** *Let  $\eta > 0$  such that  $0 < \eta < \alpha - \frac{1}{2}$ . Then*

$$\mathbf{E} \left[ \sup_{m \leq t \leq N} |\Delta J^{(m,t)}|^2 \right] \leq C |\Delta \kappa|^{2\eta}.$$

The proof is done based on some ideas from [10]. Set  $n = \lfloor t \rfloor$ . We decompose

$$J^{(m,t)}(\kappa) =: J^{(m,n)}(\kappa) + J^{(n,t)}(\kappa)$$

where  $J^{(m,n)}(\kappa) = \int_m^n e^{2i\theta_s(\kappa)} V(s) ds, \quad J^{(n,t)}(\kappa) = \int_n^t e^{2i\theta_s(\kappa)} V(s) ds.$

Our strategy is as follows: we aim at proving the estimates

$$|\Delta J^\sharp| \leq C_1 |\Delta J^{(m)}| + C_2 \sup_{m \leq t \leq N} |\Delta J^{(m,t)}| + C_3 |\Delta \kappa|^\eta, \quad (4.2)$$

where  $\sharp = (m, n)$  or  $(n, t)$  and  $C_2 = o(1)$  as  $m \rightarrow \infty$ .

#### 4.2.1 Estimate for $\Delta J^{(n,t)}$

$$\begin{aligned} \Delta J^{(n,t)} &= \frac{\omega(n)}{n^\alpha} \int_n^t \left\{ \left( e^{2i\kappa_1 s} - e^{2i\kappa_2 s} \right) e^{2i\tilde{\theta}_s(\kappa_1)} f(s-n) \right. \\ &\quad \left. + e^{2i\kappa_2 s} \left( e^{2i\tilde{\theta}_s(\kappa_1)} - e^{2i\tilde{\theta}_s(\kappa_2)} \right) f(s-n) \right\} \\ &=: \Delta J_1^{(n,t)} + \Delta J_2^{(n,t)}. \end{aligned}$$

Here we use

$$|e^{i\theta_1} - e^{i\theta_2}| \leq C_\eta |\theta_1 - \theta_2|^\eta, \quad \eta \in [0, 1], \quad (4.3)$$

which yields

$$|\Delta J_1^{(n,t)}| \leq \frac{C}{n^\alpha} n^\eta |\Delta \kappa|^\eta, \quad |\Delta J_2^{(n,t)}| \leq \frac{C}{n^\alpha} \int_n^t |\Delta \tilde{\theta}_s| ds.$$

For  $\Delta J_2$ , we estimate  $\tilde{\theta}_s$ :

$$\begin{aligned} \Delta \tilde{\theta}_s &= \frac{1}{2\kappa_1} \operatorname{Re} \int_0^s \left( e^{2i\theta_u(\kappa_1)} - 1 \right) V(u) du - \frac{1}{2\kappa_2} \operatorname{Re} \int_0^s \left( e^{2i\theta_u(\kappa_2)} - 1 \right) V(u) du \\ &= \frac{1}{2\kappa_1} \operatorname{Re} \int_0^s \left( e^{2i\theta_u(\kappa_1)} - e^{2i\theta_u(\kappa_2)} \right) V(u) du \\ &\quad + \left( \frac{1}{2\kappa_1} - \frac{1}{2\kappa_2} \right) \operatorname{Re} \int_0^s \left( e^{2i\theta_u(\kappa_2)} - 1 \right) V(u) du. \end{aligned}$$

Thus

$$|\Delta \tilde{\theta}_s| \leq C \left( |\Delta J^{(m)}| + |\Delta J^{(m,s)}| + |\Delta \kappa| |R^{(s)}(\kappa_2)| \right) \quad (4.4)$$

and therefore

$$\begin{aligned} |\Delta J_2^{(n,t)}| &\leq \frac{C}{n^\alpha} \int_n^t |\Delta \tilde{\theta}_s| ds \\ &\leq \frac{C}{n^\alpha} \left( |\Delta J^{(m)}| + \sup_{n \leq s \leq t} |\Delta J^{(m,s)}| + |\Delta \kappa| \sup_{n \leq s \leq t} |R^{(s)}(\kappa_2)| \right). \end{aligned}$$

Putting together, we have

$$\begin{aligned} |\Delta J^{(n,t)}| &\leq \frac{1}{n^{\alpha-\eta}} |\Delta \kappa|^\eta \\ &\quad + \frac{C}{n^\alpha} \left( |\Delta J^{(m)}| + \sup_{n \leq s \leq t} |\Delta J^{(m,s)}| + |\Delta \kappa| \sup_{n \leq s \leq t} |R^{(s)}(\kappa_2)| \right). \end{aligned} \quad (4.5)$$

#### 4.2.2 Estimate for $\Delta J^{(m,n)}$

We next decompose

$$\begin{aligned} J^{(m,n)}(\kappa) &= \sum_{j=m}^{n-1} \frac{\omega(j)}{j^\alpha} \int_j^{j+1} e^{2i\kappa s + 2i\hat{\theta}_s(\kappa)} f(s-j) ds \\ &\quad + \sum_{j=m}^{n-1} \frac{\omega(j)}{j^\alpha} \int_j^{j+1} e^{2i\kappa s + 2i\hat{\theta}_j(\kappa)} \left( e^{2i(\hat{\theta}_s(\kappa) - \hat{\theta}_j(\kappa))} - 1 \right) f(s-j) ds \\ &=: J_1^{(m,n)}(\kappa) + J_2^{(m,n)}(\kappa). \end{aligned}$$

The terms  $J_1^{(m,n)}(\kappa)$  and  $J_2^{(m,n)}(\kappa)$  will be estimated separately.

**Estimate on  $J_1$ .** We further decompose

$$\begin{aligned} \Delta J_1^{(m,n)} &= \sum_{j=m}^{n-1} \frac{\omega(j)}{j^\alpha} \int_j^{j+1} \left( e^{2i\kappa_1 s} - e^{2i\kappa_2 s} \right) e^{2i\hat{\theta}_j(\kappa_1)} f(s-j) ds \\ &\quad + \sum_{j=m}^{n-1} \frac{\omega(j)}{j^\alpha} \int_j^{j+1} e^{2i\kappa_2 s} \left( e^{2i\hat{\theta}_j(\kappa_1)} - e^{2i\hat{\theta}_j(\kappa_2)} \right) f(s-j) ds \\ &=: \Delta J_{1-1}^{(m,n)} + \Delta J_{1-2}^{(m,n)}. \end{aligned}$$

(1)  $\Delta J_{1-1}$  : Let

$$b_j := \int_j^{j+1} \left( e^{2i\kappa_1 s} - e^{2i\kappa_2 s} \right) e^{2i\hat{\theta}_j(\kappa_1)} f(s-j) ds.$$

Using (4.3), we have  $|b_j| \leq C_\eta |\Delta \kappa|^\eta j^\eta$ . By Proposition 3.1 with  $c_j(\omega) = C|\Delta \kappa|^\eta$ ,

$$\mathbf{E}[|J^{(m,N)}|^2] \leq C \left( \sum_{j=m}^{N-1} \frac{c \log j}{j^{\alpha-\eta}(j-c \log j)^{\alpha-\eta}} + \sum_{j=m}^{N-1} \frac{1}{j^{\alpha-2\eta-1+c\rho}} \right) |\Delta \kappa|^{2\eta}.$$

Then by Proposition 3.2 with  $c_j(\omega) = |\Delta \kappa|^\eta$ ,

$$\begin{aligned} \mathbf{E} \left[ \left| \sup_{m \leq n \leq N} \Delta J_{1-1}^{(m,n)} \right|^2 \right] &\leq C \mathbf{E} \left[ \left| \Delta J_{1-1}^{(m,N)} \right|^2 \right] + D^2 \\ &\leq C \left\{ \sum_{j=m}^{N-1} \frac{c \log j}{j^{\alpha-\eta}(j-c \log j)^{\alpha-\eta}} + \sum_{j=m}^{N-1} \frac{1}{j^{\alpha-2\eta-1+c\rho}} \right\} |\Delta \kappa|^{2\eta} + D^2 \end{aligned} \quad (4.6)$$

where

$$D := \left( \sum_{j=m}^{N-1} \frac{1}{j^{c\rho+\alpha-\eta}} + \frac{d m^\beta}{m^{\alpha-\eta}} \right) |\Delta \kappa|^\eta.$$

(2)  $\Delta J_{1-2}$  : Let

$$b_j = \int_j^{j+1} e^{2i\kappa_2 s} \left( e^{2i\hat{\theta}_j(\kappa_1)} - e^{2i\hat{\theta}_j(\kappa_2)} \right) f(s-j) ds.$$

By (4.4),

$$|b_j| \leq C |\Delta \hat{\theta}_j| \leq C \left( |\Delta J^{(m)}| + |\Delta J^{(m,j-c \log j)}| + |\Delta \kappa| |R^{(j-c \log j)}(\kappa_2)| \right).$$

Thus by Proposition 3.1 with  $\eta = 0$ , we have

$$\begin{aligned} \mathbf{E} \left[ |\Delta J_{1-2}^{(m,n)}|^2 \right] &\leq \left( \sum_{j=m}^{n-1} \frac{1}{j^{\alpha-1+c\rho}} + \frac{c \log j}{(j-c \log j)^\alpha j^\alpha} \right) \\ &\quad \times \mathbf{E} \left[ |\Delta J^{(m)}|^2 + \sup_{m \leq j \leq n} |J^{(m,j)}|^2 + |\Delta \kappa|^2 \sup_{m \leq j \leq n} |R^{(j)}|^2 \right]. \end{aligned} \quad (4.7)$$

Using Proposition 3.2 with  $\eta = 0$  and

$$c_j(\omega) = |\Delta J^{(m)}| + |\Delta J^{(m,j-c \log j)}| + |\Delta \kappa| |R^{(j-c \log j)}(\kappa_2)|,$$

yields

$$\mathbf{E} \left[ \sup_{m \leq n \leq N} |\Delta J_{1-2}^{(m,n)}|^2 \right] \leq C \mathbf{E} \left[ |\Delta J_{1-2}^{(m,N)}|^2 \right] + \mathbf{E}[|D|^2] \quad (4.8)$$



where

$$D = \left( \sum_{j=m}^{N-1} \frac{1}{j^{c\rho+\alpha-\eta}} + \frac{d m^\beta}{m^{\alpha-\eta}} \right) \sup_{m \leq j \leq N} c_j(\omega).$$

By (4.7), (4.8), we have

$$\begin{aligned} & \mathbf{E} \left[ \sup_{m \leq n \leq N} |\Delta J_{1-2}^{(m,n)}|^2 \right] \\ & \leq C \left\{ \sum_{j=m}^{N-1} \left( \frac{1}{j^{\alpha-1+c\rho}} + \frac{c \log j}{(j - c \log j)^\alpha j^\alpha} \right) + \left( \sum_{j=m}^{N-1} \frac{1}{j^{c\rho+\alpha-\eta}} \right)^2 + \left( \frac{d m^\beta}{m^{\alpha-\eta}} \right)^2 \right\} \\ & \quad \times \mathbf{E} \left[ |\Delta J^{(m)}|^2 + \sup_{m \leq j \leq N} |\Delta J^{(m,j)}|^2 + |\Delta \kappa|^2 \sup_{m \leq j \leq N} |R^{(j)}(\kappa_2)|^2 \right]. \quad (4.9) \end{aligned}$$

**Estimate on  $J_2$ .** Let  $\hat{E}_{s,j} = e^{2i(\tilde{\theta}_s(\kappa) - \hat{\theta}_j(\kappa))} - 1$ . Then we have

$$\begin{aligned} \Delta J_2^{(m,n)} &= \sum_{j=m}^{n-1} \frac{\omega(j)}{j^\alpha} \int_j^{j+1} \left\{ \left( e^{2i\kappa_1 s} - e^{2i\kappa_2 s} \right) e^{2i\hat{\theta}_j(\kappa_1)} \hat{E}_{s,j}(\kappa_1) \right. \\ & \quad + e^{2\kappa_2 s} \left( e^{2i\hat{\theta}_j(\kappa_1)} - e^{2i\hat{\theta}_j(\kappa_2)} \right) \hat{E}_{s,j}(\kappa_1) \\ & \quad \left. + e^{2i\kappa_2 s + 2i\hat{\theta}_j(\kappa_2)} \left( \hat{E}_{s,j}(\kappa_1) - \hat{E}_{s,j}(\kappa_2) \right) \right\} f(s-j) ds. \end{aligned}$$

Here we use  $|\hat{E}_{s,j}(\kappa)| \leq C j^{-\alpha} \log j$  uniformly w.r.t.  $\kappa$ . Moreover by (2.2),

$$\begin{aligned} |\Delta \hat{E}_{s,j}| &= \left| e^{2i(\tilde{\theta}_s(\kappa_1) - \hat{\theta}_j(\kappa_1))} - e^{2i(\tilde{\theta}_s(\kappa_2) - \hat{\theta}_j(\kappa_2))} \right| \\ &\leq C \left| (\tilde{\theta}_s(\kappa_1) - \hat{\theta}_j(\kappa_1)) - (\tilde{\theta}_s(\kappa_2) - \hat{\theta}_j(\kappa_2)) \right| \\ &\leq C \left\{ \frac{c \log j}{(j - c \log j)^\alpha} \cdot j^\eta \cdot |\Delta \kappa|^\eta + \frac{c \log j}{(j - c \log j)^\alpha} \sup_{j - c \log j \leq u \leq s} |\Delta \tilde{\theta}_u| \right. \\ & \quad \left. + |\Delta \kappa| \cdot \frac{c \log j}{(j - c \log j)^\alpha} \right\}. \end{aligned}$$

Substituting the above bound, we have

$$\mathbf{E} \left[ \sup_{m \leq n \leq N} |\Delta J_2^{(m,n)}|^2 \right]$$

$$\begin{aligned}
&\leq \left\{ \sum_{j=m}^{n-1} \left( \frac{\log j}{j^{2\alpha-\eta}} + \frac{c \log j}{(j - c \log j)^\alpha j^{\alpha-\eta}} + \frac{c \log j}{(j - c \log j)^\alpha j^\alpha} \right) \right\}^2 |\Delta \kappa|^{2\eta} \\
&\quad + \left\{ \sum_{j=m}^{n-1} \left( \frac{c \log j}{j^{2\alpha}} + \frac{c \log j}{(j - c \log j)^\alpha j^\alpha} \right) \right\}^2 \times \left( \mathbf{E}[|\Delta J^{(m)}|^2] \right. \\
&\quad \left. + \mathbf{E}\left[ \sup_{m \leq j \leq N} |\Delta J^{(m,j)}|^2 \right] + |\Delta \kappa|^2 \mathbf{E}\left[ \sup_{0 \leq j \leq N} |R^{(j)}(\kappa_2)|^2 \right] \right). \tag{4.10}
\end{aligned}$$

### 4.2.3 Proof of Proposition 4.2

By (4.5), (4.6), (4.9) and (4.10), we have

$$\begin{aligned}
&\mathbf{E} \left[ \sup_{m \leq n \leq N} |\Delta J^{(m,n)}|^2 \right] \leq a_m |\Delta \kappa|^{2\eta} \\
&\quad + b_m \left( \mathbf{E}[|\Delta J^{(m)}|^2] + \mathbf{E}\left[ \sup_{m \leq j \leq N} |\Delta J^{(m,j)}|^2 \right] + |\Delta \kappa|^2 \mathbf{E}\left[ \sup_{j \leq N} |R^{(j)}(\kappa_2)|^2 \right] \right)
\end{aligned}$$

where  $a_m, b_m = o(1)$ , as  $m \rightarrow \infty$ . Take  $m \gg 1$  s.t.  $b_m < 1$ . Then

$$\mathbf{E} \left[ \sup_{m \leq n \leq N} |\Delta J^{(m,n)}|^2 \right] \leq c_m |\Delta \kappa|^{2\eta} + d_m \left( \mathbf{E}[|\Delta J^{(m)}|^2] + |\Delta \kappa|^2 \mathbf{E}\left[ \sup_{j \leq N} |R^{(j)}(\kappa_2)|^2 \right] \right)$$

Here we use the fact that  $\mathbf{E}[|\Delta J^{(m)}|^2] \leq C|\Delta \kappa|^2$  (which follows from (2.3)) and Proposition 4.1, completing the proof.

## 5 Holder continuity : p-th power

In this section, we assume **A** and prove estimates on the p-th power moment of  $R$  and  $J$  using BDG inequalities.

### 5.1 Estimate on R

**Proposition 5.1** *Assume A. Then*

$$\mathbf{E} \left[ \sup_{m \leq t \leq N} |R^{(m,t)}(\kappa)|^p \right] \leq \left( \sum_{j=m}^{N-1} \frac{\mathbf{E}[\omega(j)^2]}{j^{2\alpha}} \right)^{p/2} + \left( \sum_{j=m}^{N-1} \frac{1}{j^\alpha} \right)^p.$$

*Proof.* Set  $n = \lfloor t \rfloor$ . Then

$$\begin{aligned} R^{(m,t)} &= R^{(m,n)} + R^{(n,t)} \\ R^{(n,t)} &= \int_n^t \left( e^{2i\theta_s(\kappa)} - 1 \right) V(s) ds, \quad |R^{(n,t)}| \leq Cn^{-\alpha} \end{aligned}$$

so that it suffices to estimate  $\mathbf{E}[\sup_{m \leq n \leq N} |R^{(m,n)}|^2]$ . We decompose

$$\begin{aligned} R^{(m,n)}(\kappa) &:= \sum_{j=m}^{n-1} \frac{\omega(j)}{j^\alpha} \int_j^{j+1} \left\{ e^{2i\kappa s + 2i\tilde{\theta}_j(\kappa)} \right. \\ &\quad \left. + e^{2i\kappa s + 2i\tilde{\theta}_j(\kappa)} \left( e^{2i(\tilde{\theta}_s(\kappa) - \tilde{\theta}_j(\kappa))} - 1 \right) - 1 \right\} f(s-j) ds \\ &=: R_1^{(m,n)}(\kappa) + R_2^{(m,n)}(\kappa) + R_3^{(m,n)}(\kappa). \end{aligned}$$

which is slightly different from (4.1).

(1)  $R_1^{(m,n)}(\kappa)$  :

$$\mathbf{E}[|R_1^{(n)}(\kappa)|^2] = \sum_{j=m}^{n-1} \mathbf{E} \left[ \left| \frac{\omega(j)^2}{j^{2\alpha}} \int_j^{j+1} e^{2i\kappa s + 2i\tilde{\theta}_j(\kappa)} f(s-j) ds \right|^2 \right] \leq C \sum_{j=m}^{n-1} \frac{\mathbf{E}[\omega(j)^2]}{j^{2\alpha}}.$$

By BDG,

$$\mathbf{E} \left[ \sup_{m \leq j \leq N} |R_1^{(m,j)}(\kappa)|^p \right] \leq C \mathbf{E} \left[ |R_1^{(m,N)}(\kappa)|^2 \right]^{p/2} = C \left( \sum_{j=m}^{N-1} \frac{\mathbf{E}[\omega(j)^2]}{j^{2\alpha}} \right)^{p/2}. \quad (5.1)$$

(2)  $R_2^{(m,n)}(\kappa)$  : Here we use

$$\left| e^{2i(\tilde{\theta}_s(\kappa) - \tilde{\theta}_j(\kappa))} - 1 \right| \leq 2|\tilde{\theta}_s(\kappa) - \tilde{\theta}_j(\kappa)| \leq Cj^{-\alpha}$$

and the fact that  $\omega(j)$  is bounded, yielding

$$\mathbf{E} \left[ \sup_{m \leq n \leq N} |R_2^{(m,n)}(\kappa)|^p \right] \leq C \left( \sum_{j=m}^{N-1} \frac{1}{j^{2\alpha}} \right)^p. \quad (5.2)$$

(3)  $R_3^{(m,n)}(\kappa)$  : this is similar to (1) above :

$$\mathbf{E} \left[ \sup_{m \leq j \leq N} |R_3^{(m,j)}(\kappa)|^p \right] \leq \mathbf{E} \left[ |R_3^{(m,N)}(\kappa)|^2 \right]^{p/2} \leq \left( \sum_{j=m}^{N-1} \frac{\mathbf{E}[\omega(j)^2]}{j^{2\alpha}} \right)^{p/2}. \quad (5.3)$$

By (5.1), (5.2), and (5.3), we complete the proof.  $\square$

## 5.2 Estimate on J

**Proposition 5.2** *Assume A. Then*

$$\mathbf{E} \left[ \sup_{m \leq t \leq N} |J^{(m,t)}|^p \right] \leq |\Delta \kappa|^{p\eta}, \quad p \geq 2.$$

*Proof.* Set  $n = \lfloor t \rfloor$ . Then we decompose

$$J^{(m,t)}(\kappa) =: J^{(m,n)}(\kappa) + J_0^{(n,t)}(\kappa)$$

and

$$\begin{aligned} J^{(m,n)}(\kappa) &= \sum_{j=m}^{n-1} \frac{\omega(j)}{j^\alpha} \int_j^{j+1} e^{2i\kappa s + 2i\tilde{\theta}_j(\kappa)} f(s-j) ds \\ &\quad + \sum_{j=m}^{n-1} \frac{\omega(j)}{j^\alpha} \int_j^{j+1} e^{2i\kappa s + 2i\tilde{\theta}_j(\kappa)} \left( e^{2i(\tilde{\theta}_s(\kappa) - \tilde{\theta}_j(\kappa))} - 1 \right) f(s-j) ds \\ &=: J_1^{(m,n)}(\kappa) + J_2^{(m,n)}(\kappa). \end{aligned}$$

Our strategy is to obtain inequalities similar to (4.2).

(0) Estimate for  $\Delta J_0$  : this can be done as (4.5).

$$|\Delta J_0| \leq \frac{1}{n^{\alpha-\eta}} |\Delta \kappa|^\eta + \frac{1}{n^\alpha} \left( \sup_{s \leq t} |\Delta J^{(s)}| + |\Delta \kappa| \sup_{s \leq t} |R^{(s)}| \right).$$

(1) Estimate for  $\Delta J_1$  : By the argument used in the proof of Proposition 4.2, we have

$$\begin{aligned} \mathbf{E} \left[ |\Delta J_1^{(m,N)}|^2 \right] &\leq \sum_{j=m}^{N-1} \frac{\mathbf{E}[\omega(j)^2]}{j^{2\alpha-2\eta}} \\ &\times \left( |\Delta \kappa|^{2\eta} + \mathbf{E}[|\Delta J^{(m)}|^2] + \mathbf{E} \left[ \sup_{m \leq j \leq N} |\Delta J^{(m,j)}|^2 \right] + |\Delta \kappa|^2 \mathbf{E} \left[ \sup_{m \leq j \leq N} |R^{(j)}(\kappa_2)|^2 \right] \right). \end{aligned}$$

Let  $p \geq 2$ . Taking  $p/2$ -th power on both sides,

$$\begin{aligned} \mathbf{E} \left[ |\Delta J_1^{(m,N)}|^2 \right]^{p/2} &\leq \left( \sum_{j=m}^{N-1} \frac{\mathbf{E}[\omega(j)^2]}{j^{2\alpha-2\eta}} \right)^{p/2} \left\{ |\Delta \kappa|^{p\eta} + \mathbf{E}[|\Delta J^{(m)}|^2]^{p/2} \right. \\ &\quad \left. + \mathbf{E} \left[ \sup_{m \leq j \leq N} |\Delta J^{(m,j)}|^2 \right]^{p/2} + |\Delta \kappa|^p \mathbf{E} \left[ \sup_{m \leq j \leq N} |R^{(j)}(\kappa_2)|^2 \right]^{p/2} \right\}. \end{aligned}$$

Here we use the following facts.

(i) Burkholder-Davies-Gundy inequality (BDG) :

$$\mathbf{E} \left[ |\Delta J_1^{(m,N)}|^2 \right]^{p/2} \simeq \mathbf{E} \left[ \sup_{m \leq j \leq N} |\Delta J_1^{(m,j)}|^p \right]$$

(ii) Using  $\mathbf{E}[|X|] \leq \mathbf{E}[|X|^p]^{1/p}$  valid for  $p \geq 1$ , we have

$$\mathbf{E} \left[ \sup_{m \leq j \leq N} |\Delta J^{(m,j)}|^2 \right] = \mathbf{E} \left[ \left| \sup_{m \leq j \leq N} \Delta J^{(m,j)} \right|^2 \right] \leq \mathbf{E} \left[ \left| \sup_{m \leq j \leq N} \Delta J^{(m,j)} \right|^p \right]^{2/p}.$$

Using (i), (ii) above, we have

$$\begin{aligned} \mathbf{E} \left[ \sup_{m \leq j \leq N} |\Delta J_1^{(m,j)}|^p \right] &\leq \left( \sum_{j=m}^{N-1} \frac{\mathbf{E}[\omega(j)^2]}{j^{2\alpha-2\eta}} \right)^{p/2} \left\{ |\Delta \kappa|^{p\eta} + \mathbf{E}[|\Delta J^{(m)}|^p] \right. \\ &\quad \left. + \mathbf{E} \left[ \left| \sup_{m \leq j \leq N} \Delta J^{(m,j)} \right|^p \right] + |\Delta \kappa|^p \mathbf{E} \left[ \sup_{m \leq j \leq N} |R^{(j)}(\kappa_2)|^2 \right]^{p/2} \right\}. \end{aligned} \quad (5.4)$$

(2) Estimate for  $\Delta J_2$  : by the argument in the proof of Proposition 4.2,

$$\begin{aligned} \sup_{m \leq j \leq N} |\Delta J_2^{(m,j)}|^2 &\leq \left( \sum_{j=m}^{N-1} \frac{1}{j^{2\alpha-\eta}} \right)^2 \left\{ |\Delta \kappa|^{2\eta} + |\Delta J^{(m)}|^2 \right. \\ &\quad \left. + \sup_{m \leq j \leq N} |\Delta J^{(m,j)}|^2 + |\Delta \kappa|^2 \sup_{m \leq j \leq N} |R^{(j)}(\kappa_2)|^2 \right\}. \end{aligned}$$

Take the  $p/2$ -th power, and then expectation.

$$\begin{aligned} \mathbf{E} \left[ \sup_{m \leq j \leq N} |\Delta J_2^{(m,j)}|^p \right] &\leq \left( \sum_{j=m}^{N-1} \frac{1}{j^{2\alpha-\eta}} \right)^p \left\{ |\Delta \kappa|^{p\eta} + \mathbf{E} [|\Delta J^{(m)}|^p] \right. \\ &\quad \left. + \mathbf{E} \left[ \sup_{m \leq j \leq N} |\Delta J^{(m,j)}|^p \right] + |\Delta \kappa|^p \mathbf{E} \left[ \sup_{m \leq j \leq N} |R^{(j)}(\kappa_2)|^p \right] \right\}. \end{aligned} \quad (5.5)$$

(3) Putting together : now the rest of the argument is quite similar to that of Proposition 4.2, so that we omit the details.  $\square$

## 6 Proof of Theorems

The following two propositions are the key ingredients of the proof of the clock convergence.

**Proposition 6.1** (1) Assume **A**. We then have

$$\Theta^{(n)}(c) \rightarrow c, \quad \text{for all } c \in \mathbf{R}, \quad \text{a.s.}$$

(2) Assume **B** and let  $\beta > 0$  satisfies  $2\eta\beta > 1$ . Then

$$\Theta^{(k^\beta)}(c) \rightarrow c, \quad \text{for all } c \in \mathbf{R}, \quad \text{a.s.}$$

*Proof.*

Proof of (2) : Assume **B**. Then

$$\Theta^{(N)}(c) - c = \frac{1}{2\kappa_c} \text{Re } \Delta J^{(N)} + \left( \frac{1}{2\kappa_c} - \frac{1}{2\kappa_0} \right) \text{Re } R^{(N)}(\kappa_0).$$

By Propositions 4.1, 4.2, we have

$$\begin{aligned} \mathbf{P} \left( |\Theta^{(N)}(c) - c| \geq \epsilon \right) &\leq \frac{C}{\epsilon^2} \mathbf{E} \left[ |\Delta J^{(N)}|^2 + |\Delta \kappa|^2 |R^{(N)}(\kappa_0)|^2 \right] \\ &\leq \frac{C}{\epsilon^2} \left( |\Delta \kappa|^{2\eta} + |\Delta \kappa|^2 \right) \end{aligned}$$

where we put  $\Delta \kappa = \kappa_c - \kappa_0 = \frac{\epsilon}{n}$ . Take  $\beta > 0$  such that  $2\eta\beta > 1$  and consider a subsequence of  $N$  ;  $N := k^\beta$ . Let

$$B_{k,\epsilon}(c) := \left\{ |\Theta^{(k^\beta)}(c) - c| \geq \epsilon \right\}, \quad k = 1, 2, \dots, \quad \epsilon > 0.$$

Then by the Borel–Cantelli lemma,  $\mathbf{P} \left( \limsup_{k \rightarrow \infty} B_{k,\epsilon}(c) \right) = 0$ , so that

$$\mathbf{P} \left( \bigcap_{l \geq 1} \limsup_{k \rightarrow \infty} B_{k, \frac{1}{l}}(c) \right) = 0.$$

Therefore  $\Theta^{(k^\beta)}(c) \rightarrow c$ , a.s. for any fixed  $c \in \mathbf{R}$ . Since  $h(c) := c$  is continuous and non-decreasing,

$$\Theta^{(k^\beta)}(c) \xrightarrow{k \rightarrow \infty} c$$

on the compliment of the event  $\left( \bigcup_{c \in \mathbf{Q}} \bigcap_{l \geq 1} \limsup_{k \rightarrow \infty} B_{k, \frac{1}{l}}(c) \right)$ .

Proof of (1) Assume **A**. Then Proposition 5.2 yields

$$\begin{aligned} \mathbf{P} \left( |\Theta^{(N)}(c) - c| \geq \epsilon \right) &\leq \frac{C}{\epsilon^p} \mathbf{E} \left[ |\Delta J^{(N)}|^p + |\Delta \kappa|^p |R^{(N)}(\kappa_0)|^p \right] \\ &\leq \frac{C}{\epsilon^p} (|\Delta \kappa|^{p\eta} + |\Delta \kappa|^p). \end{aligned}$$

Taking  $p \gg 1$  s.t.  $p\eta > 1$  would give us the a.s. convergence without taking further subsequence.  $\square$

**Proposition 6.2** *For any fixed  $\kappa$ ,*

$$\tilde{\theta}_n(\kappa) \xrightarrow{n \rightarrow \infty} \tilde{\theta}_\infty(\kappa).$$

*Proof.* Since  $\tilde{\theta}_n(\kappa) = \frac{1}{2\kappa} Re R^{(n)}(\kappa)$ , it suffices to show the convergence of  $R^{(n)}(\kappa)$ . By Proposition 4.1,

$$\begin{aligned} \mathbf{E} \left[ \sup_{2^k \leq n \leq 2^{k+1}} |R^{(2^k, n)}(\kappa)|^2 \right] &\leq C \sum_{2^k \leq j \leq 2^{k+1}} \frac{1}{j^{2\alpha-\epsilon}} + \left( \sum_{2^k \leq j \leq 2^{k+1}} \frac{1}{j^{2\alpha-\epsilon}} \right)^2 + \frac{1}{2^{2(\alpha-\eta-\beta)k}} \\ &\leq \frac{C}{2^{(2\alpha-1-\epsilon)k}} \end{aligned}$$

for  $\epsilon > 0$  sufficiently small, by taking  $\eta + \beta$  sufficiently small. By Chebyshev's inequality,

$$\mathbf{P} \left( \sup_{2^k \leq n \leq 2^{k+1}} |R^{(2^k, n)}(\kappa)|^2 \geq \frac{1}{k^4} \right) \leq C \frac{k^4}{2^{(2\alpha-1-\epsilon)k}}.$$

By the Borel-Cantelli lemma, with probability one

$$\sup_{2^k \leq n \leq 2^{k+1}} |R^{(2^k, n)}(\kappa)| \leq \frac{1}{k^2}$$

for  $k \gg 1$ , implying that  $\{R^{(n)}(\kappa)\}_n$  is Cauchy.  $\square$

**Lemma 6.3** ([8], Lemma 3.3) *Let  $\Psi_n, n = 1, 2, \dots$ , and  $\Psi$  are continuous and increasing functions such that  $\lim_{n \rightarrow \infty} \Psi_n(x) = \Psi(x)$  pointwise. If  $y_n \in \text{Ran} \Psi_n$ ,  $y \in \text{Ran} \Psi$  and  $y_n \rightarrow y$ , then it holds that*

$$\Psi_n^{-1}(y_n) \xrightarrow{n \rightarrow \infty} \Psi^{-1}(y).$$

*Proof of Theorems 1.1, 1.2*

In the representation of the Laplace transform of  $\xi_n$  (Lemma 2.1), we use Propositions 6.1, 6.2, and Lemma 6.3.  $\square$

For the proof of Theorem 1.3, for given  $n$ , rearrange the eigenvalues  $\{\kappa_k(n)\}$  of  $H_n$  such that

$$\cdots < \kappa'_0(n) < \kappa_0 < \kappa'_1(n) < \cdots$$

**Lemma 6.4** *For any fixed  $k$ ,  $\kappa'_k(n) = \kappa_0 + o(1)$ ,  $n \rightarrow \infty$ .*

*Proof.* By definition of  $\kappa'_k(n)$ ,

$$(m_n + k)\pi = \theta_n(\kappa'_k(n)) = \kappa'_k(n)n + \tilde{\theta}_n(\kappa'_k(n)) \quad (6.1)$$

Write  $\kappa_0 n = m_n \pi + \beta_n$ ,  $m_n \in \mathbf{N}$ ,  $\beta_n \in [0, \pi)$ . Substituting  $m_n \pi = \kappa_0 n - \beta_n$  into (6.1) yields

$$\kappa'_k(n) = \kappa_0 + \frac{-\beta_n + k\pi - \tilde{\theta}_n(\kappa'_k(n))}{n}$$

By (2.2),  $n^{-1}\tilde{\theta}_n(\kappa'_k(n)) \rightarrow 0$ .  $\square$

*Proof of Theorem 1.3*

$$\begin{aligned} \theta_n(\kappa'_j(n)) &= [\theta_n(\kappa_0)]_\pi + j\pi \\ \theta_n(\kappa'_{j+1}(n)) &= [\theta_n(\kappa_0)]_\pi + (j+1)\pi \end{aligned}$$

from which we have

$$(\kappa'_{j+1}(n) - \kappa'_j(n))n - (\tilde{\theta}_n(\kappa'_{j+1}(n)) - \tilde{\theta}_n(\kappa'_j(n))) = \pi.$$

Here we note that, by Proposition 5.2, the family  $\{J^{(N)}(\kappa)\}$  is tight as continuous function-valued process. Hence  $\tilde{\theta}_n(\kappa) \rightarrow \tilde{\theta}_\infty(\kappa)$  locally uniformly w.r.t.  $\kappa$ . Since  $\kappa'_k(n) \rightarrow \kappa_0$  by Lemma 6.4, we have  $\tilde{\theta}_n(\kappa'_k(n)) \rightarrow \tilde{\theta}_\infty(\kappa_0)$ .  $\square$



## 7 Deterministic potentials

### 7.1 Symbolic dynamical systems

Let  $\mathcal{A} = (a_1, \dots, a_M)$  be an abstract finite set ("alphabet"), and consider the probability spaces  $(\Omega, \mathcal{F}, \mathbf{P})$ ,  $(\Omega_+, \mathcal{F}_+, \mathbf{P}_+)$  where

$$\Omega = \mathcal{A}^{\mathbf{Z}}, \quad \Omega_+ = \mathcal{A}^{\mathbf{N}},$$

$\mathcal{F}$  (resp.,  $\mathcal{F}_+$ ) is the sigma-algebra generated by the cylinder subsets

$$C_{i_1, \dots, i_k}(A_1, \dots, A_k) := \{\omega : \omega_{i_j} \in A_j, j = 1, \dots, k\}$$

(in the case of  $\mathcal{F}_+$ , the indices  $i_j$  are non-negative), and  $\mathbf{P}$  (resp.,  $\mathbf{P}_+$ ) is a probability measure on  $\mathcal{F}$  (resp., on  $\mathcal{F}_+$ ) invariant under the shift endomorphism (isomorphism, in the case of  $\Omega$ )  $T$  defined by

$$(T\omega)_i = \omega_{i+1}.$$

For brevity, below we often write  $\Omega_\bullet$ ,  $\mathcal{F}_\bullet$  etc., where  $\bullet$  is "nothing" or "+". In all cases, we keep the same notation for the shift transformation  $T$ .

In a number of interesting applications, the pair  $(\mathbf{P}_\bullet, T)$  is markovian, i.e.,  $\mathbf{P}_\bullet$  is a Markov measure w.r.t.  $T$ :

$$\mathbf{P}\{\omega_{t+1} = a \mid \mathcal{F}_{\leq t}\} = \mathbf{P}\{\omega_{t+1} = a \mid \mathcal{F}_{=t}\}$$

where  $\mathcal{F}_{\leq t}$  (resp.,  $\mathcal{F}_{=t}$ ) is generated by the values of the symbols  $\omega_i$  with  $i \leq t$  (resp.,  $i = t$ ). Equivalently, for any  $a, b, b_{-1}, \dots \in \mathcal{A}$ , one has

$$\mathbf{P}\{\omega_{t+1} = a \mid \omega_t = b, \omega_{t-1} = b_{-1}, \dots\} = \mathbf{P}\{\omega_{t+1} = a \mid \omega_t = b\} = \Pi_{ab},$$

for some stochastic matrix  $\Pi = (\Pi_{ab})$ , with  $\sum_a \Pi_{ab} = 1$ . A particular subclass of Markov systems is formed by the Bernoulli shifts, where  $\mathbf{P}_\bullet$  is a product measure  $\mathbf{P}_\bullet = \mu^{\mathbf{Z}_\bullet}$ , and  $\mu$  is a probability measure on  $\mathcal{A}$  endowed with the maximal sigma-algebra containing all singletons  $\{a\}$ ,  $a \in \mathcal{A}$ .

### 7.2 Symbolic representations for some hyperbolic systems

#### 7.2.1 Dyadic expansion of the unit circle

Here  $\Omega = \mathbf{T}^1 = \mathbf{R}/\mathbf{Z}$ , and we identify it, as a measure (in fact, probability) space with the interval  $[0, 1)$  endowed with the Haar (Lebesgue, in this case)

measure  $\mathbf{P}$ . Consider the measurable transformation  $T : \Omega \rightarrow \Omega$  defined by

$$T : x \mapsto \{2x\} \equiv 2x \pmod{1}.$$

The Lebesgue measure is  $T$ -invariant: for any measurable subset  $A \subset \Omega$ ,  $\mathbf{P}T^{-1}A = \mathbf{P}A$ . It suffices to check the latter identity for the intervals  $A = [x, y)$  where it is obvious, since

$$T^{-1}[x, y) = \left[\frac{x}{2}, \frac{y}{2}\right) \cup \left[\frac{x}{2} + \frac{1}{2}, \frac{y}{2} + \frac{1}{2}\right),$$

and each of the two disjoint intervals in the above RHS has length  $(y - x)/2$ . Naturally,  $T$  is only an endomorphism, but not isomorphism, for it is not invertible, so it generates a semi-group  $\{T^t, t \in \mathbf{Z}_+\}$ .

The standard symbolic representation for this dynamical system is obtained with the help of the binary expansion of the real numbers  $x \in [0, 1)$ ,

$$x = \sum_{i=0}^{\infty} \frac{\omega_i}{2^{i+1}},$$

so the identification  $x$  with the infinite word  $(\omega_0, \omega_1, \dots) \in \{0, 1\}^{\mathbf{Z}_+}$  is a bijection, if one excludes the words having an infinite tail of the form  $(\dots, \omega_n, 1, 1, 1, \dots)$ , using the identity

$$\sum_{i=n+1}^{\infty} 2^{-i-1} = 2^{-n-1}.$$

For example,  $(0, 1, 1, 1, \dots)$  and  $(1, 0, 0, 0, \dots)$  are two dyadic representations of the number  $\frac{1}{2}$ . We only define the transformations defined (and, where applicable, invertible) Lebesgue-a.e.

It is straightforward that  $T$  becomes the left shift on the set of the words  $\omega = (\omega_0, \omega_1, \dots)$ .

### 7.2.2 Baker's transform

Baker's transform, or baker's map (N.B.: here "baker" is not a family name but merely a profession) is a particular realization of the Bernoulli shift considered in Sect. 7.2.1. From the symbolic dynamics point of view, it is obtained from the dyadic expansion of the circle by a canonical procedure

extending an endomorphism (with time given by a semi-group  $\mathbf{N} = \mathbf{Z}_+$ ) to an isomorphism (invertible measure-preserving transformation with time  $\mathbf{Z}$ ). Curiously, the geometrical realization is quite simple:  $T = \mathcal{C} \circ \mathcal{E}$ , where

$$\mathcal{E}(x, y) = (2x, y/2),$$

so that  $\mathcal{E}([0, 1]^2) = [0, 2] \times [0, \frac{1}{2}]$ , and

$$\mathcal{C}(x, y) = \begin{cases} (x, y), & (0 \leq x < 1) \\ (x - 1, y + \frac{1}{2}), & (\text{otherwise}). \end{cases} \quad (7.1)$$

The second stage consists, geometrically, in cutting the rectangle  $[0, 2] \times [0, \frac{1}{2}]$  into two halves, then leaving  $[0, 1] \times [0, \frac{1}{2}]$  invariant and putting  $[1, 2] \times [0, \frac{1}{2}]$  on top of the first rectangle. To obtain a symbolic dynamics representation  $T_{\mathcal{A}}$  of  $T$ , use the dyadic expansions

$$x = \sum_{i=0}^{+\infty} \omega_i 2^{-i-1}, \quad y = \sum_{i=1}^{+\infty} \omega_{-i} 2^{-i}$$

and set

$$\Phi : (x, y) \mapsto (\dots, \omega_{-2}, \omega_{-1}, \omega_0, \omega_1, \dots).$$

Then  $T_{\mathcal{A}} = \Phi \circ T \circ \Phi^{-1}$  is the left shift on infinite words  $\omega \in \{0, 1\}^{\mathbf{Z}}$ . Indeed, on the  $x$ -coordinate  $T = \mathcal{C} \circ \mathcal{E}$  acts exactly as the dyadic extension, since  $\mathcal{E}$  acts so, while  $\mathcal{C}$  adds to the  $x$ -coordinate either 0 or  $-1 = 0 \pmod{1}$ . The dyadic digits of  $y$ , shift to the left, for  $\mathcal{E}$  is multiplication by  $1/2$  in the  $y$ -direction; this determines all digits of the image  $T(x, y)$  with negatives indices except the place no.  $(-1)$ . As to this symbol, the definition (7.1) clearly shows that it equals 0 if  $x < 1/2$ , i.e., if  $\omega_0 = 0$ , and 1 otherwise, so in both cases it is given by  $\omega_0$ .

It is readily seen that  $\omega_0(x, y)$ , as function of the phase point  $\mathbf{u} = (x, y) \in \mathbf{T}^2$ , is merely the indicator function of the rectangle  $C_0 := [0, \frac{1}{2}] \times [0, 1]$ . Respectively,

$$\omega_t(\mathbf{u}) = 1_{C_0}(\mathbf{u}), \quad t \in \mathbf{Z}.$$

Equivalently, introducing the partition  $\Omega = C_0 \sqcup C_1$ ,  $C_1 = \Omega \setminus C_0$ , one can identify the word  $(\omega_t(\mathbf{u}), t \in \mathbf{Z})$  with the sequence of the ordinal numbers of the partition elements visited by the trajectory  $\{T^t \mathbf{u}\}$ . Since  $T$  shrinks

the vertical coordinate  $y$  by the factor  $1/2$ , and  $T^{-1}$  does the same to the horizontal coordinate  $x$ , the cylinder sets

$$\bigcap_{t=-n}^n \{\mathbf{u} : T^t \mathbf{u} \in C_{a_t}\}, \quad a_t \in \{0, 1\},$$

have exponentially decaying diameter as  $n \rightarrow \infty$ .

### 7.2.3 Algebraic automorphisms of tori

In the general case  $\nu \geq 2$ , the construction of Markov partitions for hyperbolic toral automorphisms was proposed by Sinai [16]. This construction is rather technical and particularly tedious for the tori of dimension  $\nu > 2$ , so we give only an upshot in the case  $\nu = 2$ , and refer the interested reader to the original paper [16] and to the books on ergodic theory, e.g., [2, 4, 7, 13, 15].

Consider a unimodular matrix with integer entries

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{Z}),$$

thus having eigenvalues  $(\lambda_1, \lambda_2) = (\lambda, \lambda^{-1})$ , and assume that  $|\lambda| > 1$ , so the modulus of the second eigenvalue is smaller than 1. The most famous example is

$$M = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad \lambda_{1,2} = \frac{3 \pm \sqrt{5}}{2}. \quad (7.2)$$

Since  $M$ , acting in  $\mathbf{R}^2$  by multiplication, maps the lattice  $\mathbf{Z}^2 \hookrightarrow \mathbf{R}^2$  into itself, it also acts on the factor-space  $\mathbf{R}^2/\mathbf{Z}^2 = \mathbf{T}^2$ . In the case (7.2) it is usually called Arnold's Cat Map. The inequalities  $|\lambda^{-1}| < 1 < |\lambda|$  mean that  $M$  is hyperbolic: it has an extending and contracting eigenspaces. An astute geometrical procedure allows one to partition the torus into a finite union of parallelepipeds  $C_a$ ,  $a \in \mathcal{A} = \{a_1, \dots, a_N\}$ , with sides parallel to the extending and contracting eigenspaces, in such a way that

- for a.e.  $\mathbf{u} = (x, y) \in \mathbf{T}^2$ , the sequence of symbols  $(a_{k(t)}, t \in \mathbf{Z})$  such that

$$T^t \mathbf{u} \in C_{a_{k(t)}}, \quad t \in \mathbf{Z},$$

determines the point  $\mathbf{u}$  uniquely; in other words, the torus point  $\mathbf{u}$  is identified with the sequence of the ordinal numbers of the parallelepipeds it visits under the dynamics  $\{T^t\}$ ;

- under the above identification, the Lebesgue measure on  $\mathbf{Z}^2$  corresponds to a Markov measure w.r.t. the shift  $T$ : writing  $\mathbf{u} \leftrightarrow (\dots, \omega_{-1}, \omega_0, \omega_1, \dots)$ , one has

$$\mathbf{P}\{\omega_{t+1} = a \mid \omega_t = b, \omega_{t-1} = b_{-1}, \dots\} = \mathbf{P}\{\omega_{t+1} = a \mid \omega_t = b\} = \Pi_{ab},$$

for some irreducible stochastic matrix  $\Pi$ .

Such a partition is called a Markov partition for the dynamical system  $(\Omega, \mathcal{F}, \mathbf{P}, T)$ .

### 7.3 Local regularity, quasi-locality and decay of correlations

For our purposes, the key feature of the Markov partitions is exponential decay of the diameter of a cylinder set in the torus

$$\mathcal{X}_{(-n, \dots, n)}(\alpha_{-n}, \dots, \alpha_n) = \{x(\omega) \in \mathbf{T}^\nu : \omega_i = \alpha_j, -n \leq i \leq n\}$$

as  $n \rightarrow +\infty$ . The geometrical mechanism of this decay is essentially the same as for the baker's map, although the decay exponent is determined by the eigenvalues of the generating linear map  $\mathcal{L}$ . Therefore, for any two points  $x, y$  whose symbolic representations ("letters"  $\omega_i(x)$  and  $\omega_i(y)$ ) agree on a long interval of indices,  $i \in \{-n - n + 1, \dots, n\}$ , we have

$$\text{dist}_{\mathbf{T}^\nu}(x, y) \leq q^n, \quad q \in (0, 1) . \quad (7.3)$$

Consequently, for any continuous function  $f : \mathbf{T}^\nu \rightarrow \mathbf{R}$  with continuity modulus

$$s_f(\epsilon) := \sup_{y \in B_\epsilon(x)} |f(y) - f(x)|$$

one has for the points  $x, y$  satisfying (7.3)

$$|f(y) - f(x)| \leq s_f(q^n) .$$

In particular, for any Hölder continuous function of order  $\beta \in (0, 1]$ , we have

$$|f(y) - f(x)| \leq C\tilde{q}^n, \quad \tilde{q} = q^\beta \in (0, 1).$$

In all considered examples, the existence of a Markov partition gives rise to the quasi-locality of the deterministic random potentials as functions of symbols in the infinite words  $(\omega_i)_{i \in \mathbf{Z}_\bullet}$ .

Introduce the following notation: for a function  $f : \mathcal{A}^{\mathbf{Z}_\bullet} \rightarrow \mathbf{R}$  and an index subset  $I \subset \mathbf{Z}_\bullet$ , let

$$\text{Var}_I(f) := \sup_{\omega', \omega : \pi_I \omega' = \pi_I \omega} |f(\omega') - f(\omega)|, \quad (7.4)$$

where  $\pi_I(\omega)$  is the finite sub-word of  $\omega$  formed by the letters  $(\omega_i, i \in I)$ .

**Definition 7.1** *A function  $f : \mathcal{A}^{\mathbf{Z}_\bullet} \rightarrow \mathbf{R}$  is called quasi-local if there exists  $C = C(f) > 0$  and  $q \in (0, 1)$  such that for any  $n \geq 1$  and any finite word  $(\omega_{-n}, \omega_{-n+1}, \dots, \omega_n)$*

$$\text{Var}_{[-m, n]}(f) \leq Cq^{m \wedge n}. \quad (7.5)$$

In turn, the quasi-locality implies exponential decay of correlations (this decay may be slower for the sampling functions  $f$  featuring lower regularity than Hölder continuity).

The bottom line is that in the above mentioned examples of hyperbolic dynamical systems on tori  $\mathbf{T}^\nu \cong [0, 1)^\nu \subset \mathbf{R}^\nu$ , the corresponding deterministic potentials feature a fast decay of correlations sufficient for the extension of the Kolmogorov's connection between the convergence in mean square and the a.s. convergence of the random series

$$\sum_{j=1}^{\infty} \frac{v_j}{j^\alpha}, \quad \alpha \in \left(\frac{1}{2}, 1\right].$$

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